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Markoyian Multiserver Vacation Models Quasi - Birth - And - Death Process-Short Communication

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Abstract

In this section we consider a Quasi-Birth-and-Death Process. A QBD process is the generalization of a birth-and-death process from a one-dimensional state space to a multidimensional state space. It can be analyzed by using the matrix analytical method (Neuts, 1981 and Latouche, 2011).

Key words: Multiserver, infinitesimal generator, positive recurrent.

Introduction

Multidimensional Markov process arise in many fields of science, engineering and business. In many practical queuing systems multiple servers attend to the queue like in call centre, bank, restaurant etc. QBD has played a central role in computational probability. A QBD Markov chain in two dimensional process where the first dimension is called the level and second the phase (G. Latouche).

Consider a two-dimensional markov process $\{(x(t), y(t), t \geq 0)\}$ with state space $\Omega = \{(i, j) : i \geq 0, 1 \leq j \leq n\}$.

The process $\{(x(t), y(t), t \geq 0)\}$ is called a QBD process if the infinitesimal generator of the process is given by

$$Q = \begin{bmatrix} A_0 & C_0 & & & \\ B_1 & A_1 & C_1 & & \\ & B_2 & A_2 & C_2 & \\ & & B_3 & A_3 & C_3 \\ & & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot \end{bmatrix} \quad (1)$$

Where all submatrices are $n \times n$ matrices $A_i, i \geq 0$, have negative diagonal elements and non negative off-diagonal elements and $C_i, i \geq 0$ and $B_i, i \geq 1$, are all non-negative matrices satisfying

$$(A_0 + C_0)e = (B_i + A_i + C_i)e = 0 \quad i \geq 1$$

State set $\{(0,1), \dots, (0,n)\}$ is said to be the boundary level; state set $\{(i,1), \dots, (i,n)\}$ is said to be level i . In many applications, we have a special case of equation (1) where the non-boundary submatrices of the infinitesimal generator are independent of level i ^{1,3}. Thus Q is written as

$$Q = \begin{bmatrix} A_0 & C_0 & & & \\ B_1 & A & C & & \\ & B & A & C & \\ & & B & A & C \\ & & \cdot & \cdot & \cdot \end{bmatrix} \quad (2)$$

Assume that the QBD process is positive recurrent, and let (X, Y) be the limit of $\{X(t), Y(t)\}$ as $t \rightarrow \infty$. Denote the stationary probabilities by

$$\pi_i, j = P\{X = i, Y = j\} = \lim_{t \rightarrow \infty} P\{X(t) = i, Y(t) = j\} \quad (i, j) \in \Omega$$

$$\pi_i = (\pi_{i1}, \pi_{i2}, \dots, \pi_{in}), i \geq 0$$

so we present the following theorem.

Theorem 1: The irreducible QBD process is positive recurrent if and only if the matrix equation.

$$R^2B + RA + C = 0 \quad (3)$$

has the minimum nonnegative solution R , with spectral radius $\text{sp}(R) < 1$, and a set of linear homogenous equations

$$\pi_0 = (A_0 + RB_1)e = 0$$

has the positive solution, furthermore, the stationary distribution can be expressed as the matrix geometric form

$$\pi_i = \pi_0 R^i, i \geq 0$$

When π_0 is the positive solution of the set of linear homogeneous equations and satisfies the normalization condition

$$\pi_0(I-R)^{-1}e = 1$$

Theorem 2 : The irreducible QBD process with Q^* is positive recurrent if and only if the matrix equation (3) has the minimum nonnegative solution R , with the spectral radius $sp(R) < 1$, and the $n_1 + n$ linear homogeneous equations below have the positive solution³.

$$(\pi_0, \pi_1)B[R] = 0$$

Where $B[R]$ is the $(n_1+n) \times (n_1+n)$ matrix

$$B(R) = \begin{bmatrix} A_0 & C_0 \\ B_1 & A + RB \end{bmatrix}$$

Furthermore, the stationary distribution can be expressed as the matrix geometric form

$$\pi_i = \pi_1 R^{i-1}, \quad i \geq 1$$

and (π_0, π_1) satisfies the normalization condition as

$$\pi_0 e + \pi_1 (I-R)^{-1} e = 1$$

The stationary distribution $\pi_i, i \geq 0$ is called the modified matrix geometric solution. To analyze the M/M/C queue with vacations, we encounter a more complex variant of the process; for a system with c servers. Not only the $i = 0$ level but also $i = 1, 2, 3, \dots, C-1$ level becomes boundary levels and these boundary levels contain different numbers of states. Thus we denote the number of states at level i by $n_i, 0 \leq i \leq c-1$. The infinitesimal generator is given by

$$Q = \begin{bmatrix} A_0 & C_0 & & & & \\ B_1 & A_1 & C_1 & & & \\ & \dots & \dots & & & \\ & & & B_{C-1} & A_{C-1} & C_{C-1} \\ & & & & B_C & A & C \\ & & & & & B & A & C \\ & & & & & & B & A \\ & & & & & & & \ddots & \ddots \\ & & & & & & & & \ddots & \ddots \end{bmatrix}. \quad (4)$$

Where A_i are the $m_i \times m_i$ matrices $0 \leq i \leq c-1$; B_i are the $m_i \times m_{i-1}$ matrices, $0 \leq i \leq c-1$; C_i are $m_i \times m_{i+1}$ matrices, $0 \leq i \leq c-2$ C_{c-1} and B_c are the $n_{c-1} \times n$ and $n \times n_{c-1}$ matrices respectively; and A, B and C are all $n \times n$ square matrices. We assume that the QBD process is positive recurrent. The stationary distribution is denoted by

$$\begin{aligned} \pi_i &= (\pi_{i1}, \pi_{i2}, \dots, \pi_{in_i}) & 0 \leq i \leq c-1 \\ \pi_i &= (\pi_{i1}, \pi_{i2}, \dots, \pi_{in}) & i \geq c \end{aligned}$$

The matrix Q in equation (4) be repartitioned into the block from that is consistent with Q^* as follows:
Let

$$A_0 = \begin{bmatrix} A_0 & C_0 & & & \\ B_1 & A_1 & C_1 & & \\ & \cdot & \cdot & & \\ & \cdot & \cdot & & \\ & & B_{C-2} & A_{C-2} & C_{C-2} \\ & & & B_{C-1} & A_{C-1} \end{bmatrix}$$

$$B_1 = (0, B_C), \quad C_0 = \begin{pmatrix} 0 \\ C \\ C-1 \end{pmatrix}$$

Where A_0 is the square matrix of order $n^* = n_0 + n_1 + \dots + n_{C-1}$ and B_1 and C_0 are the $n \times n^*$ and $n^* \times n$ matrices, respectively. These equation (4) can be written as

$$Q = \begin{bmatrix} A_0 & C_0 & & & \\ B_1 & A & C & & \\ & B & A & C & \\ & & B & A & C \\ & & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot \end{bmatrix}$$

and this QBD process becomes a variant of the canonical form.

Theorem 3 : The irreducible QBD process is positive recurrent if and only if the matrix equation

$$R^2B + RA + C = 0$$

has the minimum nonnegative solution, R with the spectral radius $\text{sp}(R) < 1$, and the linear homogeneous equations

$$(\pi_0, \pi_1, \dots, \pi_{C-1}, \pi_C)B[R] = 0 \quad (5)$$

has a positive solution where

$$B[R] = \begin{bmatrix} A_0 & C_0 & & & \\ B_1 & A_1 & C_1 & & \\ & \cdot & \cdot & & \\ & \cdot & \cdot & & \\ & & & B_{C-1} & A_{C-1} & C_{C-1} \\ & & & & B_C & A + RB \end{bmatrix} \quad (6)$$

Furthermore, the stationary distribution can be expressed as the matrix geometric form

$$\pi_i = \pi_C R^{i-c}, \quad i \geq c \quad (7)$$

Where $(\pi_0, \pi_1, \dots, \pi_{C-1}, \pi_C)$ is the positive solution of equation (5) and satisfies the normalization condition

$$\sum_{i=0}^{c-1} \pi_i e + \pi_C (I - R)^{-1} e = 1$$

Conclusion

In the last, the relevance of matrix analytical method (MAM) in analysing the QBD process is demonstrated taking into account the positive recurrent irreducible QBD process. The stationary distribution is also indicated as the matrix geometric form.

Scope of future work:

The QBD model is quite general and can be used for many stochastic models in different areas of research and work. Like production lines, Telecommunication systems, management problems etc. The QBD process has central importance in computational probability, the knowledge of which can be further enhanced for utilization in different fields. This area is full of prospects for further researchers.

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