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Zero Divisor Conjecture for Group Semifields

W.B. VASANTHA KANDASAMY[1]

[1]Department of Mathematics, Indian Institute of Technology,
Chennai- 600 036 (India)
E-mail: vasanthakandasamy@gmail.com

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Abstract

In this paper the author for the first time has studied the zero divisor conjecture for the group semifields; that is groups over semirings. It is proved that whatever group is taken the group semifield has no zero divisors that is the group semifield is a semifield or a semidivision ring. However the group semiring can have only idempotents and no units. This is in contrast with the group rings for group rings have units and if the group ring has idempotents then it has zero divisors.

Key words: Group semirings, group rings semifield, semidivision ring.

1. Introduction

Here just we recall the zero divisor conjecture for group rings and the conditions under which group rings have zero divisors. Here rings are assumed to be only fields and not any ring with zero divisors. It is well known[1,2,3] the group ring KG has zero divisors if G is a finite group. Further if G is a torsion free non abelian group and K any field, it is not known whether KG has zero divisors. In view of the problem 28 of [1]. However in this paper it is proved whatever be the group G, the group semiring SG where S is a semifield has no zero divisors. This paper has three sections. Section one is introductory in nature. Section two studies zero divisors in group semifields and section three gives conclusions based on our study.

For notions of group semirings refer[8-11]. For more about group rings refer[1-6]. Throughout this paper the semirings used are semifields.

2. Zero divisors in group semifields :

In this section zero divisors in group semifields SG of any group G over the semifields ($Z^+ \cup \{0\}$, $R^+ \cup \{0\}$, $Q^+ \cup \{0\}$, $\langle Z^+ \cup I \rangle \cup \{0\}$, $\langle R^+ \cup I \rangle \cup \{0\}$, $\langle Q^+ \cup I \rangle \cup \{0\}$ and distributive lattices L such that $a_i \cap a_j \neq 0$ for $a_i, a_j \in L \setminus \{0\}$) are studied. It is proved SG is a semifield if G is commutative and SG is a semidivision ring if G is non-commutative.

Just for the sake of completeness of this paper the definition of group semifield is recalled.

Definition 2.1: Let S be a semifield and G any group. SG = {Collection of all finite formal sums of the form $\sum_i s_i g_i$*, i runs over a finite index;* $s_i \in S$ *and* $g_i \in G$*} is defined as the group semifield of the group G over the semifield S if the following conditions are satisfied.*

(i) $\sum_{i=1}^{n} \alpha_i g_i = \sum_{i=1}^{n} \beta_i g_i$ *if and only if* $\alpha_i = \beta_i$ *for i = 1, 2, …, n.*

(ii) $\left(\sum_{i=1}^{n} \alpha_i g_i\right) + \left(\sum_{i=1}^{n} \beta_i g_i\right) = \sum_{i=1}^{n} (\alpha_i + \beta_i)\, g_i$

(iii) $\left(\sum_{i=1}^{n} \alpha_i g_i\right)\left(\sum_{i=1}^{n} \beta_i g_i\right) = \sum_{k=1}^{t} \gamma_k m_k$ *where*

$\gamma_k = \sum_{i=1}^{n} \alpha_i \beta_j$; $m_k = g_i h_j$.

(iv) $r_i\, g_i = g_i\, r_i$ *for all* $r_i \in S$ *and* $g_i \in G$.

(v) $s \sum_{i=1}^{n} s_i g_i = \sum_{i=1}^{n} s s_i g_i$ *for* $r \in S$ *and* $\sum_{i=1}^{n} s_i g_i \in SG$.

SG is a semiring/semifield with 0 as the additive identity. As $1 \in S$ *and* $1 \in G$. $1.G \subseteq SG$ *and* $S.1 \subseteq SG$.

First we will illustrate this situation by some examples. For more about semifields refer[11].

Example 2.1. Let $S = R^+ \cup \{0\}$ be the semifield of reals. $G = \langle g \mid g^{24} = 1 \rangle$ be the cyclic group of order 24. SG be the group semifield of finite order.

SG has no zero divisors. SG has no units other than $x = 1.g^i$ where $g^i \in G$; which is known as the trivial units. SG has idempotents.

For take $\alpha = \frac{1}{2}(1 + g^{12}) \in SG$.

$$\alpha^2 = \left(\frac{1}{2}(1 + g^{12})\right)^2 = \frac{1}{4}(1 + 2g^{12} + g^{24})$$

$$= \frac{1}{4}(2 + 2g^{12})$$

$$= \frac{1}{2}(1 + g^{12})$$

$$= \alpha.$$

However in this group semiring the presence of this idempotent does not lead to zero divisors as in case of rings.

$\beta = \frac{1}{4}(1 + g^6 + g^{12} + g^{18})$ is again an idempotent of SG.

$\gamma = \frac{1}{3}(1 + g^8 + g^{16}) \in SG$ is an idempotent of SG.

$\delta = \frac{1}{8}(1 + g^3 + g^6 + g^9 + g^{12} + g^{15} +$

$g^{18} + g^{21}) \in SG$ is an idempotent.

$$y = \frac{1}{12}\ (1 + g^2 + g^4 + \ldots + g^{22}) \in SG$$

is an idempotent of SG.

$$\eta = \frac{1}{6}\ (1 + g^4 + g^8 + g^{12} + g^{16} + g^{20})$$

is an idempotent of SG.

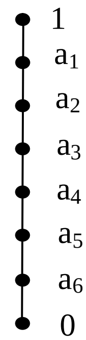
However SG is only a semifield as SG is commutative.

Note: In the above example if the real semifield $R^+ \cup \{0\}$ is replaced by the rational semifield $Q^+ \cup \{0\}$, the above conclusions are true.

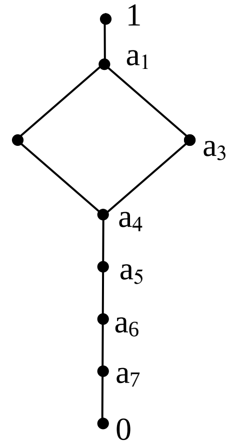
But however if the real semifield $R^+ \cup \{0\}$ is replaced by the integer semifield $Z^+ \cup \{0\}$ then the group semifield has no idempotents. Interested reader can replace $R^+ \cup \{0\}$ by $\langle R^+ \cup I\rangle \cup \{0\}$, $\langle Q^+ \cup I\rangle$ $\{0\}$ and $\langle Z^+ \cup I\rangle \cup \{0\}$ and study this problem; it is left as a matter of routine. However $\langle Z^+ \cup I\rangle \cup \{0\}$ has no idempotents other than $I^2 = I$.

Next one example in which a distributive lattice of finite order which is a semifield is used. We say a distributive lattice with 0 and 1. Lattice L is said to have a zero divisor if $a_i \cap a_j = 0$ where $a_i, a_j \in L \setminus \{0\}$. Such distributive lattices are declared as semifields which has no zero divisors. So in this paper, finite a distributive lattice L with 0 and 1 will be termed as semifields of finite order. First we will give one or two examples of them. The distributive lattices with the Hasse diagram which are semifields is as follows:

L_1=

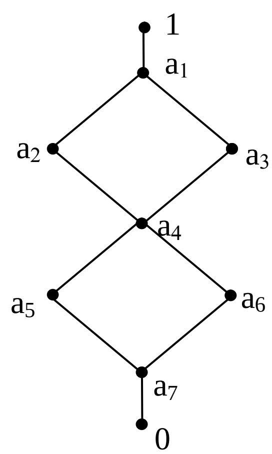


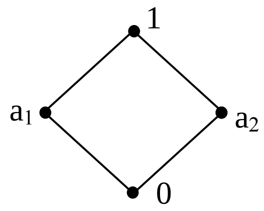
L_2 =

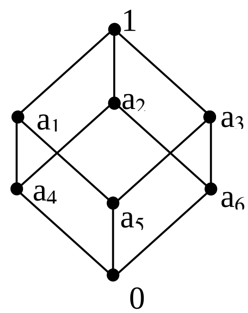


Consider,

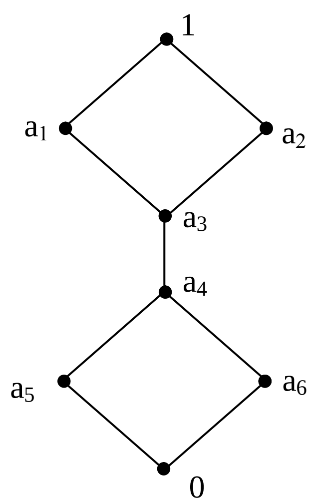
L_3 =



$L_4 =$ 

$L_5 =$ 

and

$L_6 =$ 

These three distributive lattices are only semirings as they contain zero divisors.

Those group semiring using the semirings L_4 or L_5 or L_6 will be a semiring with zero divisors.

Example 2.2: Let $G = S_7$ be the symmetric group of degree seven. $L = C_{10}$ be the chain lattice $0 < a_8 < a_7 < \ldots < a_1 < 1$. LG be the group semifield. LG is a non-commutative semiring. LG has no zero divisors and units. However LG has idempotents. LG is a semidivision ring of finite order.

In view of all these the following observations are made.

The group semifields SG of all groups G over every semifield S (finite semifields or otherwise) has no zero divisors only trivial units and idempotents which are nontrivial.

However if $S = Z^+ \cup \{0\}$, the semifield of integers; SG the group semifield where G is any group. SG has no idempotents, non trivial units and zero divisors.

Theorem 2.1: Let G be any group finite or infinite commutative or non-commutative. S be any semifield, SG the group semifield. SG has no zero divisors and SG is a semifield.

Proof: Follows from the fact S has no zero divisors and S is a strict semiring so SG cannot have zero divisors as $\alpha.\beta = 0$ is impossible for any $\alpha, \beta \in SG \setminus \{0\}$.

Corollary: If in the above theorem S is taken as a semiring with zero divisors. For every group G, SG, the group semiring has zero divisors.

Proof is direct.

Next a brief comparison is made in the following section.

3. Conclusions

It is proved SG the group semifield of any group G over any semifield S is a semifield or a semidivision ring. Secondly when SG the group semifield has idempotents still SG has no zero divisors. This is in sharp contrast between the group rings RG where R is a field and G any group has zero divisors if G is a finite group or a torsion group. Thus the famous open conjecture for group rings has no relevance in case of group semifields.

Finally another big contrast is existence of idempotents does not lead to zero divisors in group semifields however in group rings the presence of idempotents guarantees the presence of zero divisors.

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