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Some generating functions of modified Jacobi polynomials by Lie algebraic method

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Abstract

A good number of generating functions involving Jacobi, modified Jacobi polynomials have been derived by various researchers using Weisner's group-theoretic method. In this paper, we have studied $P_n^{(\alpha+n,\beta)}(x)$, a different modification of Jacobi polynomials by the same method of Weisner and obtained some novel results by suitable interpretations of the index, n and the parameter, α simultaneously.

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1. Introduction

The Jacobi polynomial, $P_n^{(\alpha,\beta)}(x)$, defined[1] by:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} \; {}_2F_1\left[\begin{matrix} -n \;; & 1+\alpha+\beta+n \;; & \\ & 1+\alpha \;; & \end{matrix} \frac{1-x}{2}\right] \tag{1}$$

is a solution of the following ordinary differential equation:

$$(1-x^2)\frac{d^2y}{dx^2} + [\beta-\alpha-(2+\alpha+\beta)x]\frac{dy}{dx} + (1+\alpha+\beta+n)\,y = 0. \tag{2}$$

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In this paper we consider $P_n^{(\alpha+n,\beta)}(x)$, a modification of $P_n^{(\alpha,\beta)}(x)$ satisfying the following ordinary differential equation:

$$(1-x^2)\frac{d^2y}{dx^2}\ [\beta - \alpha - n - (2+\alpha+\beta+n)\,x]\ \frac{dy}{dx} + n\,(1+\alpha+\beta+2n)\,y = 0. \qquad (3)$$

The aim at writing this paper is to investigate some novel generating functions of $P_n^{(\alpha+n,\beta)}(x)$, a modification of the Jacobi polynomials by using Weisner's[2-4] group-theoretic method which is lucidly presented in the monograph "obtaining generating functions" by E.B. McBride[5]. For previous works on Jacobi polynomials, one may refer to the works[6-17]. The main results of this investigation are given in section-3.

2 Group-Theoretic Discussion:

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, α by $y\frac{\partial}{\partial y}$, n by $z\,\frac{\partial}{\partial z}$ and $P_n^{(\alpha+n,\beta)}(x)$ by by u (x,y,z) in (3), we get the following partial differential equation:

$$[(1-x^2)\frac{\partial^2}{\partial x^2} + \{\beta - x(\beta+2)\}\frac{\partial}{\partial x} - y(1+x)\frac{\partial^2}{\partial y\,\partial x} + yz\frac{\partial^2}{\partial z\,\partial y} + z(3+\beta)\frac{\partial}{\partial z} - (1+x)z\frac{\partial^2}{\partial z\,\partial x} + 2z^2\frac{\partial^2}{\partial z^2}]\ u(x,y,z) = 0. \quad (4)$$

Thus, $u_1(x,y,z) = P_n^{(\alpha+n,\beta)}(x)\,y^\alpha\,z^n$ is a solution of (4), since $P_n^{(\alpha+n,\beta)}(x)$ is a solution of (3).

Let us now seek two first order linear partial differential operators A_3 and A_4 such that

$$A_3\left[P_n^{(\alpha+n,\beta)}(x)\,y^\alpha\,z^n\right] = a\,(\alpha,n)\,P_{n-1}^{(\alpha+n+1,\beta)}(x)\,y^{\alpha+2}\,z^{n-1} \quad (5)$$

and

$$A_4\left[P_n^{(\alpha+n,\beta)}(x)\,y^\alpha\,z^n\right] = b\,(\alpha,n)\,P_{n+1}^{(\alpha+n-1,\beta)}(x)\,y^{\alpha-2}\,z^{n+1}. \quad (6)$$

Using (5) and the relation (18):

$$(1+x)\frac{d}{dx}\left(P_n^{(\alpha,\beta)}(x)\right) = n\,P_n^{(\alpha,\beta)}(x) + (\beta+n)\,P_{n-1}^{(\alpha+1,\beta)}(x) \qquad (7)$$

We easily obtain

$$A_3 = (1+x)\,y^2\,z^{-1}\,\frac{\partial}{\partial x} - y^2\,\frac{\partial}{\partial z}$$

Such that

$$A_3\left[P_n^{(\alpha+n,\beta)}(x)\ y^\alpha\ z^n\right] = (\beta+n)\,P_{n-1}^{(\alpha+n+1,\beta)}(x)\,y^{\alpha+2}\ z^{n-1}.$$

Similarly using (6) and the relation[18]:

$$\frac{d}{dx}\left(P_n^{(\alpha,\beta)}(x)\right) = \frac{1}{1-x^2}\,[(n+\alpha+\beta+1)(x-1)+2\alpha]\,P_n^{(\alpha,\beta)}(x) - 2\,(n+1)P_{n+1}^{(\alpha-1,\beta)}(x), \quad (8)$$

We obtain

$$A_4 = (1-x^2)\,y^{-2}\,z\,\frac{\partial}{\partial x} - (x+1)\,y^{-1}\,z\,\frac{\partial}{\partial y} - 2\,x\,y^{-2}\,z^2\,\frac{\partial}{\partial z} - (1+\beta)(x-1)\,y^{-2}\,z,$$

Such that

$$A_4\left[\,P_n^{(\alpha+n,\beta)}(x)\,y^{\alpha}\,z^{n}\,\right] = -2(n+1)\,P_{n+1}^{(\alpha+n-1,\beta)}(x)\,y^{\alpha-2}\,z^{n+1}.$$

To find the group of operators, let us write $A_1 = y\,\frac{\partial}{\partial y}$, $A_2 = z\,\frac{\partial}{\partial z}$.

Then we have the following commutator relations:

$[A_1, A_2] = 0$; $[A_1, A_3] = 2A_3$;
$[A_1, A_4] = -2A_4$; $[A_2, A_3] = -A_3$;
$[A_2, A_4] = A_4$; $[A_3, A_4] = -2\,[2A_2 + 1 + \beta]$,

where

$$[A, B] = AB - BA\,.$$

From the above commutator relations we can easily state the following theorem:

Theorem: The set of operators $\{1, A_i : i = 1,2,3,4\}$, where 1 stands for the identity operator, generates a Lie algebra $\mathcal{L}$.

Now the partial differential operator L, given by

$$L = (1-x^2)\,\frac{\partial^2}{\partial x^2} + \{\beta - x(\beta+2)\}\,\frac{\partial}{\partial x} - y\,(1+x)\,\frac{\partial}{\partial y\,\partial x} - (1+x)\,z\,\frac{\partial^2}{\partial z\,\partial x} + z\,(3+\beta)\,\frac{\partial}{\partial z} + y\,z\,\frac{\partial^2}{\partial y\,\partial z} + 2\,z^2\,\frac{\partial^2}{\partial z^2}$$

Can be expressed as follows:

$$(1+x)\,L = A_4A_3 + 2A_2^2 + 2\,\beta\,A_2\,. \quad (9)$$

It can be easily verified that $(1+x)\,L$ commutes with $\{1, A_i : i = 1,2,3,4\}$ *i.e.*

$$[(1+x)L,\ A_i] = 0, \quad i = 1,2,3,4. \quad (10)$$

The extended form of the groups generated by A_i (i=1,2,3,4) are given below:

$$e^{a_1A_1}\ f(x,y,z) = f(x,\ e^{a_1}y,\ z) \quad (11)$$

$$e^{a_2A_2}\ f(x,y,z) = f(x,\ y,\ e^{a_2}z) \quad (12)$$

$$e^{a_3A_3}\ f(x,y,z) = f\left(\frac{z\,x + y^2\,a_3}{z - y^2\,a_3},\ y,\ z - y^2a_3\right) \quad (13)$$

$$e^{a_4A_4}\ f(x,y,z) = [1-(1-x)\,y^{-2}\,z\,a_4]^{-\beta-1}\, f\left(\frac{x + (1-x)\,y^{-2}\,z\,a_4}{1-(1-x)\,y^{-2}\,z\,a_4},\ \frac{y\,(1-2\,y^{-2}\,z\,a_4)}{1-(1-x)\,y^{-2}z\,a_4},\ \frac{z\,(1-2\,y^{-2}z\,a_4)}{\{1-(1-x)\,y^{-2}\,z\,a_4\}^2}\right) \quad (14)$$

Where all a_i (i=1,2,3,4) are arbitrary constants.

Now from the above, we have

$$e^{a_4A_4}\ e^{a_3A_3}\ e^{a_2A_2}\ e^{a_1A_1}\ f(x,y,z) = [1-(1-x)y^{-2}\,z\,a_4]^{-\beta-1}$$

$$\times f\left(\frac{z\{x+(1-x)y^{-2}z\ a_4\}+y^2 a_3(1-2y^{-2}z a_4)\{1-(1-x)y^{-2}z a_4\}}{\{z-y^2 a_3(1-2y^{-2}z\ a_4)\}\ \ \{1-(1-x)y^{-2}\ z\ a_4\}},\right.$$
$$\left.\frac{e^{a_1}y(1-2y^{-2}z\ a_4)}{1-(1-x)y^{-2}\ z\ a_4},\ \frac{e^{a_2}(1-2y^{-2}\ z\ a_4)\{z-y^2\ a_3\ (1-2y^{-2}\ z\ a_4)\}}{\{1-(1-x)y^{-2}z\ a_4\}^2}\right). \qquad (15)$$

3 Generating Functions:

From the above discussion, we see that $P_n^{(\alpha+n,\beta)}(x)\ y^\alpha\ z^n$ is a solution of the systems:

$L\ u = 0$ $\qquad$ $L\ u = 0$ $\qquad$ $L\ u = 0$

$(A_1 - \alpha)\ u = 0$; $\qquad$ $(A_2 - n)\ u = 0$; $\qquad$ $(A_1 + A_2 - \alpha - n)\ u = 0$.

From (10), we easily get

$$S\big((1+x)L\big)\ \left[P_n^{(\alpha+n,\beta)}(x)\ y^\alpha\ z^n\right] = ((1+x))L\ S\left[P_n^{(\alpha+n,\beta)}(x)\ y^\alpha\ z^n\right] = 0,$$

where

$$S = e^{a_4A_4}\ e^{a_3A_3}\ e^{a_2A_2}\ e^{a_1A_1}.$$

Therefore, $S\left[P_n^{(\alpha+n,\beta)}(x)\ y^\alpha\ z^n\right]$ is annulled by $(1+x)\,L$.

By setting $a_1 = a_2 = 0$ in (15) and replacing $f(x,y,z)$ by $P_n^{(\alpha+n,\beta)}(x)\ y^\alpha\ z^n$, we get

$$e^{a_4A_4}\ e^{a_3A_3}\left[P_n^{(\alpha+n,\beta)}(x)\ y^\alpha\ z^n\right]$$
$$= [1-(1-x)y^{-2}za_4]^{-\beta-1-\alpha-2n}y^\alpha(1-2y^{-2}za_4)^{\alpha+n}z^n(1-a_3y^2z^{-1}+2a_3a_4)^n$$
$$\times P_n^{(\alpha+n,\beta)}\left(\frac{z\{x+(1-x)y^{-2}z\ a_4\}+\ y^2\ a_3(1-2y^{-2}\ z\ a_4)\ \{1-(1-x)y^{-2}\ z\ a_4\}}{\{z-y^2\ a_3\ (1-2y^{-2}\ z\ a_4)\}\ \ \{1-(1-x)y^{-2}\ z\ a_4\}}\right).$$

On the otherhand

$$e^{a_4A_4}\ e^{a_3A_3}\left[P_n^{(\alpha+n,\beta)}(x)\ y^\alpha\ z^n\right]$$
$$= \sum_{r=0}^{\infty}\sum_{k=0}^{r+n}\frac{(-2\ a_4)^r}{r!}\ \frac{(-a_3)^k}{k!}\ (-\beta-n)_k\ (n-k+1)_r$$
$$\times P_{n-k+r}^{(\alpha+n+k-r,\beta)}(x)\quad y^{\alpha+2k-2r}\quad z^{n-k+r}.$$

Equating the above two results, we get

$$[1-(1-x)y^{-2}za_4]^{-\beta-1-\alpha-2n}(1-2y^{-2}za_4)^{\alpha+n}(1-a_3y^2z^{-1}+2a_3a_4)^n$$
$$\times P_n^{(\alpha+n,\beta)}\left(\frac{z\{x+(1-x)y^{-2}z\ a_4\}+\ y^2\ a_3(1-2y^{-2}\ z\ a_4)\ \{1-(1-x)y^{-2}\ z\ a_4\}}{\{z-y^2\ a_3\ (1-2y^{-2}\ z\ a_4)\}\ \ \{1-(1-x)y^{-2}\ z\ a_4\}}\right)$$
$$= \sum_{r=0}^{\infty}\sum_{k=0}^{r+n}\frac{(-2\ a_4)^r}{r!}\ \frac{(-a_3)^k}{k!}\ (-\beta-n)_k\ (n-k+1)_r\times P_{n-k+r}^{(\alpha+n+k-r,\beta)}(x)\quad y^{2k-2r}\quad z^{-k+r}. \qquad (16)$$

Which does not seem to have appeared in the earlier works. Now we consider the following three cases:

Case 1: By setting $a_3=1$; $a_4=0$ and putting $y^2z^{-1}=-t$ in (16), we get

$$(1+t)^n\ P_n^{(\alpha+n,\beta)}\left(\frac{x-t}{1+t}\right)=\sum_{k=0}^{n}\frac{(-\beta-n)_k}{k!}\ P_{n-k}^{(\alpha+n+k,\beta)}(x)\ t^k\ . \qquad (17)$$

Case 2: By setting $a_3=0$; $a_4=1$ and putting $2\,y^2z=t$ in (16), we get

$$\left\{1+(1-x)\ \frac{t}{2}\right\}^{-1-\alpha-\beta-2n}\ (1+t)^{\alpha+n}\ \ P_n^{(\alpha+n,\beta)}\left(\frac{x-\ (1-x)\ \frac{t}{2}}{1+\ (1-x)\ \frac{t}{2}}\right)$$

$$=\sum_{r=0}^{\infty}\frac{(n+1)_r}{r!}\ \ P_{n+r}^{(\alpha+n-r,\beta)}(x)\ t^r\ . \qquad (18)$$

Sub Case : Putting $n=0$ in (18), we get

$$\left\{1+(1-x)\ \frac{t}{2}\right\}^{-1-\alpha-\beta}\ (1+t)^{\beta}=\sum_{r=0}^{\infty}P_r^{(\alpha-r,\beta)}(x)\ t^r\ .$$

Finally, using the symmetry relation [1]:

$$P_n^{(\beta,\alpha)}(-x)=\ (-1)^n\ P_n^{(\alpha,\beta)}(x),$$

and then simplifying, we get

$$\left\{1-(1+x)\ \frac{t}{2}\right\}^{-1-\alpha-\beta}\ (1-t)^{\beta}=\sum_{r=0}^{\infty}P_r^{(\alpha,\beta-r)}(x)\ t^r\ ,$$

which is well known Feldhim's[19] formula and is also derived by W. A. Alsalam[20], V. K. Verma[21] by different methods.

Case 3: By setting $a_3=-\frac{1}{w}$; $a_4=1$ and putting $y^2\ z^{-1}=t^{-1}$ in (16), we get

$$(1-2\ t)^{\alpha+n}\left\{t+\frac{1}{w}(1-2\ t)\right\}^n\ \{1-t\,(1-x)\}^{-\beta-1-\alpha-2\,n}$$

$$\times\ P_n^{(\alpha+n,\beta)}\left(\frac{t\,\{x+\ (1-x)\ t\}-\ \frac{1}{w}\ (1-2\ t)\,\{1-\ (1-x)\ t\}}{\left\{t+\frac{1}{w}\ (1-2\ t)\right\}\ \{1-\ (1-x)\ t\}}\right)$$

$$=\sum_{r=0}^{\infty}\sum_{k=0}^{r+n}\frac{(-2\,)^r}{r!}\ \frac{\left(\frac{1}{w}\right)^k}{k!}\ (-\beta-n)_k\ \ (n-k+1)_r\ \ \times P_{n-k+r}^{(\alpha+n+k-r,\beta)}(x)\qquad t^{n-k+r}\ .$$

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