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On Properties of Hyperdistribution of Real Hypersurfaces of Almost Hyperbolic Hermitian Manifold

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Abstract

The purpose of the present paper is to study properties of Pseudo-conformal mapping hyperdistribution of real hypersurfaces of almost hyperbolic Hermitian manifold.

1. Introduction

Definition: Let us consider a differentiable manifold M of class C^∞ endowed with a tensor field F of type (1, 1) such that

$\tilde{F}^2 = I$, i.e. . . .or $\tilde{F}_k^h \tilde{F}_i^k = \delta_i^h$,

and g (FX, FY) + g(X, Y) = 0

Then we say that g is compatible with structure F and (F, g) is called almost hyperbolic Hermitian structure and the manifold M with this structure is called almost hyperbolic Hermitian manifold.

Summation Convention: In the sequel, manifolds, tensor fields, connections and mappings we consider are assumed to be differentiable and of class C^∞ unless otherwise stated and the indices a, b, c, d, e, $\cdots$ run over the range {1, 2, $\cdots$ 2n + l}, the summation convention being used with respect to this system of indices.

Let there be given, on a manifold M of odd dimension 2n + l (≥3), a tensor field f of type (1, 1), a vector field ξ and a 1-form θ satisfying

(1.1) $f^2 = I - \theta \otimes \xi$, $f(\xi)=0$, $\theta(f)=0$, $\theta(\xi) = 1$,

I being the identity tensor field of type (1, 1), or

$$(1.2)\qquad f_e^a f_b^e = \delta_b^a - \theta \otimes \xi,$$

$$f_e^a \xi^e = 0, \quad \theta_e f_b^e = 0, \quad \theta_e \xi^e = 1,$$

f_b^a, ξ^a and θ_b denoting components of f, ξ and θ respectively. Then the triple (f, ξ, θ) is called an almost para contact structure in M.

We define tensor fields S of type (1, 2), G of type (0, 2), T of type (0, 2), P of type (1, 1) and Q of type (0, 1) as those with components

$$(1.3)\quad S_{cb}^a = f_c^e \nabla_e f_b^a - f_b^e \nabla_e f_c^a - (\nabla_c f_b^e - \nabla_b f_c^e) f_e^a + (\nabla_c \theta_b - \nabla_b \theta_c)\xi^a,$$

$$(1.4)\quad G_{cb} = f_c^e (\nabla_e \theta_b - \nabla_b \theta_e),$$

(1.5) $T_{cb} = G_{cb} - G_{bc}$,

(1.6) $P_b^a = -[\xi^e \nabla_e f_b^a - (\nabla_e \xi^a) f_b^e + (\nabla_b \xi^e) f_b^a]$,

(1.7) $Q_b = -[\xi^e \nabla_e \theta_b + (\nabla_b \xi^e)\theta_e]$,

respectively, where ∇ denotes the operator of covariant differentiation with respect to an arbitrary symmetric affine connection in M. We easily see that these tensor fields are independent of the symmetric connection used to define them. Then S and G are respectively called the torsion tensor and the Levi tensor of (f, ξ, θ). The following propositions are well known[4]:

(A_1) S = 0 implies T = 0, P = 0 and Q = 0;

(A_2) P = 0 implies Q = 0.

When the tensor field S vanishes identically, the almost para contact structure (f, ξ, θ) is said to be normal.

We now state an elementary lemma for later use. Let V be a vector space over real number field with almost hyperbolic Hermitian structure F. That is, F: V → V is a linear transformation satisfying $F^2 = I$ and g (FX, FY) + g(X, Y) = 0. Then V is necessarily even-dimensional, say dimV = 2n + 2(≥4). Take arbitrarily a (2n + l) -dimensional subspace W of V. Then FW is also (2n + l)-dimensional. We can now state

Lemma 1.1: Put D = W∩ FW and N = D - FW. Then FD = D, FN⊂W, V = W + FW, dim D = 2n, N= {ax_o + y/ a∈R, a≠O, y∈D), x_0 being a fixed element of N, and any element x of N is uniquely represented as x = ax_o + y (a∈R, y∈D).

The subset N appearing in Lemma 1.1 has two connected components, each of which is homeomorphic to a Euclidean space of dimension 2n + l. The subset T is called the affine normal space to W in the vector space V with almost hyperbolic Hermitian structure F.

2. Hypersurfaces of almost hyperbolic Hermitian manifold:

Let M be a almost hyperbolic Hermitian manifold of real dimension 2n + 2(≥4) with almost hyperbolic Hermitian structure F, where F is a tensor field of type (1, 1) in M satisfying $\tilde{F}^2 = I$, i.e. . . .

(2.1a) $\tilde{F}_k^h \tilde{F}_i^k = \delta_i^h$,

and

(2.1b) g (FX, FY) + g(X, Y)=0, $\tilde{F}_i^h$ denoting components of $\tilde{F}$.

Let there be given a hypersurface M immersed in $\tilde{M}$. For each point P of M, denote the tangent space to $\tilde{M}$ and that to M at P by $T_P(\tilde{M})$ and $T_P(M)$ respectively. Then the subspace $D_P = T_P\{M) \cap \tilde{F}\, T_P(M)$ is 2n-dimensional and hence the correspondence P →D_P defines a distribution D of dimension 2n in M. Since FD = D, we can define a tensor field J of type (1, 1) in D by JX = $\tilde{F}$ X, X being an arbitrary vector field belonging to D. Then $\tilde{F}^2$ = I implies $J^2 = I_D$, where I_D denotes the identity tensor field of type (1, 1) in D. Thus the D is called a hyperdistribution with almost hyperbolic Hermitian structure J in M and said to be induced in M from $\tilde{F}$ by the immersion[3].

Since the tangent space $T_P(\tilde{M})$ is a vector space with almost hyperbolic Hermitian structure $\tilde{F}$, by Lemma 1.1 the subspace $T_P(M)$ of $T_P(\tilde{M})$ has its affine normal space N_P. We call $N = \bigcup_{P \in M} N_P$ the affine normal bundle to the hypersurface M.

Since N_P has two connected components, each of which is homeomorphic to a Euclidean space, N has a global cross-section if M is orientable.

Let $\bar{U}$ be a coordinate neighborhood of $\tilde{M}$ such that any connected component U of $\bar{U} \cap M$ is a coordinate neighborhood of M. In the sequel by U we mean such a coordinate neighborhood of M. Take a local cross-section C of the affine normal bundle N over U and call it a local affine normal to M in U. Then by Lemma 1.1 $\tilde{F}$ C is tangent to M in U and hence

(2.2) $\xi = \tilde{F} C$

is a non-vanishing vector field in U. Next, for any vector field X in M, we can decompose $\tilde{F}$ X uniquely as

(2.3) $\tilde{F} X = fX + \theta(X) C,$

where fX is tangent to M. Thus f and θ are a tensor field of type (1, 1) and a 1-form in U respectively. Applying $\tilde{F}$ to (2.3) and using $\tilde{F}^2 = I$, we find $X = (f^2X + \theta(X)\xi) + \theta(fX)$ C, which implies

(2.4) $f^2 = I - \theta \otimes \xi, \qquad \theta(f) = 0.$

If we put $X = \xi$ in (2.3), we obtain $\tilde{F} = f(\xi) + \theta(\xi)$ C. On the other hand (2.2) gives $\tilde{F}\xi = C$. Hence we get

(2.5) $f(\xi) = 0, \quad \theta(\xi) = l.$

Equations (2.4) and (2.5) show that the triple (f, ξ, θ) is an almost contact structure in U, which is called an almost contact structure induced in M by an affine normal C in U. A vector field X in M belongs to D if and only if $\tilde{F}$ X belongs to D. Thus, because of (2.3), X belongs to D if and only if $\theta(X) = 0$. Hence the distribution D is defined by $\theta = 0$ in U. Therefore the almost contact structure (f, ξ, θ) is associated with the hyperdistribution D with complex structure e [3-9].

We now take another affine normal C to M in U. Then by Lemma 1.1 we have

(2.6) $\bar{C} = -\frac{1}{\alpha}(C + A),$

where α is a non-vanishing function and A a vector field being tangent to M and belonging to D, α and A being defined in U. Thus we have[8-14]

(2.7) $\bar{f} = -f + \theta \otimes \xi, \quad \bar{\xi} = -\frac{1}{\alpha}(\xi - fA),$

$\bar{\theta} = \alpha\theta,$

where ($\bar{f}$, $\bar{\xi}$, $\bar{\theta}$) is the almost contact structure induced in M by (2.3) and (2.5), C being replaced by $\bar{C}$. The change (2.7) of almost contact structures has been discussed in[3] and is called a change of almost contact structures associated with D[4-7].

3. Induced affine connections :

We now assume that the ambient manifold $\tilde{M}$ is a complex manifold of complex dimension n + l (≥ 2) with almost hyperbolic Hermitian structure $\tilde{F}$. It is well known that there is a symmetric affine connection $\tilde{\nabla}$ satisfying $\tilde{\nabla}\,\tilde{F} = 0$, i. e.

(3.1) $$\tilde{\nabla}_j\, \tilde{F}_i^{\,h} = 0$$

In the sequel we fix this affine connection $\tilde{\nabla}$.

Consider a real hypersurface M immersed in $\tilde{M}$ and a coordinate neighborhood U of M such that U is a connected component of $\bar{U} \cap M$, $\bar{U}$ being a coordinate Neighborhood of $\tilde{M}$. Let (x^h) and (y^a) be coordinates in $\bar{U}$ and in U respectively[7-11].

We assume that M is represented in $\bar{U}$ by

(3.2) $$x^h = x^h(y^a).$$

Take an affine normal C to M in U and put

(3.3) $$B_b^{\,h} = \partial x^h / \partial y^a$$

in U. Then $B_b = B_b^{\,h}\partial/\partial x^h$ and $C = C^h \partial/\partial x^h$ form an affine (2n+2)-frame along U.

Thus on putting $\begin{pmatrix} B_i^a \\ C_i \end{pmatrix} = -\left(B_b^h, C^h\right)^{-1}$

we have

(3.4) $$B_b^i\, B_i^a = \delta_b^a\,,\quad B_i^a\, C^i = 0\,,\quad C_i C^i = 1\,;$$

(3.5) $$B_e^h B_i^e + C^h C_i = -\,\delta_i^h\,.$$

Thus $B^a = B_i^a \partial x^i$ and $C = C_i \partial x^i$ form a coframe dual to {B_b, C} along U.

The affine connection ∇ induced in U from $\tilde{T}_{ji}^{\,h}$ with respect to the affine normal C has, by definition, components given by

(3.6) $$T_{cb}^{\,a} = \left(\partial_c B_b^h + \tilde{T}_{ji}^{\,h} B_c^j B_b^i\right) B_h^a$$

where $\partial_b = \partial / \partial y^b$ and $\tilde{T}_{ji}^{\,h}$ denote components of $\tilde{\nabla}$ in $\tilde{U}$. Since $\tilde{\nabla}$ is symmetric, i.e . . . . since $\tilde{T}_{ji}^{\,h} = \tilde{T}_{ij}^{\,h}\nabla$ is also symmetric, i.e. . . . $T_{cb}^{\,a} = T_{bc}^{\,a}$. Thus if we define the so- called vander Waerden- Bortolotti covariant derivative of B_b^h along M by

(3.7) $$\nabla_c B_b^h = -\partial_c B_b^h - \tilde{T}_{ji}^{\,h} B_c^j B_b^i + T_{cb}^{\,a} B_a^h$$

in U, then we have $\left(\nabla_c B_b^h\right) B_b^a = 0$ 0, which shows that $\nabla_c B_b^h$ is of the form

(3.8) $$\nabla_c B_b^h = h_{cb} C^h\,,$$

where h_{cb} are defined by

(3.9) $h_{cb} = h_{bc} = \partial_c B_b^h + \left(\tilde{T}_{ji}^{\,h} B_c^j B_b^i\right) C_h$

and are called components of the covariant second fundamental tensor h of M with respect to the affine normal C, h being of type (0, 2).

Differentiating $B_b^h B_h^a = -\delta_b^a$ covariantly along M and using (3.8) and $C_h B_h^a = 0$, we find $B_{b_{cb}}^h \left(\nabla_c B_h^a\right) = 0$, from which

(3.10) $\nabla_c B_i^a = -H_c^{\,a} C_i$

where $\nabla_c B_i^a$ are defined by

(3.11) $\nabla_c B_i^a = -\partial_c B_i^a - \tilde{T}_{ji}^{\,h} B_c^j B_h^a - T_{cb}^{\,a} B_i^b$

in U and $H_c^{\,a}$ by

(3.12) $H_c^{\,a} = \left(\partial_c B_b^h + \tilde{T}_{ji}^{\,h} B_c^j C^i\right) B_h^a$

The $H_c^{\,a}$ are called components of the mixed second fundamental tensor H of M with respect to the affine normal C in U, H being of type (1.1).

We next differentiate $B_i^a C^i = 0$ covariantly along M and use (3.10). Then we obtain $H_c^{\,a} - B_i^a \left(\nabla_c C^i\right) = 0$ from which

(3.13) $\nabla_c C^h = H_c^{\,a} B_a^h - l_c C^h$,

where l_c are defined by

(3.14) $l_c = \left(\partial_c C^h + \tilde{T}_{ji}^{\,h} B_c^j C^i\right) C_h$

and $\nabla_c C^h$ by

(3.15) $\nabla_c C^h = \partial_c C^h + \tilde{T}_{ji}^{\,h} B_c^j C^i$

in U. The l_c are called components of the third fundamental tensor *l* of M with respect to the affine normal C in U, *l* being of type (0, 1). The *l* gives a linear connection in the one-dimensional vector bundle $\bigcup_{p \in U} \{aC_p / a \in R\}$ over U.

Finally, differentiating $B_b^i C_i = 0$ covariantly along M and using (3.8), $C^i B_i^a = 0$ and $C^i C_i = 1$, we find $l_c - C^i \left(\partial_c C_i\right) = 0$, from which

(3.16) $\nabla_c C_i = h_{ce} + l_c C_i$ where $\nabla_c C_i$ are defined in U by

(3.17) $\nabla_c C_i = \partial_c C_i + \tilde{T}_{ji}^{\,h} B_c^j C_h$.

Equations (3.8) and (3.10) are those of Gauss for the real hypersurface M and equations (3.11) and (3.13) are those of Weingarten for M.

Consider a vector field $X = X^h \partial / \partial x^h$ tangent to M. Then we have $X^h = X^a B_a^h$

Thus using (3.8), we have

(3.18) $\nabla_c X^h = -\left(\partial_c X^a\right) B_a^h - h_{cb} X^b C^h$,

where we have put in U, $\nabla_c X^h = \partial_c X^h$

$$+\tilde{T}_{ji}^{\ h} B_c^j X^i\ ,\ \nabla_c X^a = \partial_c X^a + T_{cb}^a X^b$$

Let (f, ξ, θ) be the e almost contact structure induced in M by the affine normal C to M in U. The n (2.2) and (2.3) can be written as

(3.19) $\tilde{F}_i^{\ h} C^i = \xi^b B_b^h$,

(3.20) $\tilde{F}_i^{\ h} B_b^i = f_b^a B_a^h - l_b C^h$

respectively. Applying ∇_c to (3.20) and using $\nabla_c \tilde{F}_i^{\ h} = B_c^j \tilde{\nabla}_j \tilde{F}_i^{\ h} = 0$, we obtain

$$h_{cb}\left(f^a B_a^h\right) = \left(\nabla_c f_b^a\right) B_a^h + h_{ce} f_b^e C^h - \left(\nabla_c \theta_b\right) C^h + \left(H_c^a B_a^h - l_c C^h\right),$$

where we have used (2.2), (2.3) with $X = B_b$ (3.16) and (3.17). Thus we obtain

(3.21) $\nabla_c f_b^a\ h_{cb} \xi^a - H_c^a \theta_b$,

(3.22) $\nabla_c \theta_b = h_{ce} f_b^e + l_c \theta_b$.

Next, applying ∇_c to (3.19), we have in a similar way as above from which

(3.23) $\nabla_c \xi^a = f_e^a H_c^e - l_c \xi^a$,

(3.24) $H_c^e \theta_e = -h_{ce} \xi^e$.

Substituting (3.21), (3.22) and (3.23) into (1.3) and using (3.24), we obtain

(3.25) $S_{cb}^a = (-H_e^a f_c^e + f_e^a H_c^e - l_c \xi^a)\theta_b$

$+ (-H_e^a f_b^e + f_e^a H_b^e - l_b \xi^a)\theta_c$,

(3.26) $G_{cb} = h_{cb} + f_c^e\ f_b^d\ h_{ed} - h_{be}\ \xi^e\ \theta_c + f_c^e\ l_e\ \theta_b$

(3.28) $Q_b = -l_b + (l_e\ \xi^e)\ \theta_b + h_{ed}\ \xi^e\ f_b^d$.

When a hyperdistribution D with almost hyperbolic Hermitian structure J is given on a manifold of odd dimension and when $S_{cb}^a \equiv 0$, (mod θ_c, θ_b) is satisfied for an almost para contact structure (f, ξ, θ) associated with D, the D is said to be torsionless . Thus we have from (3.25).

Proposition 3.1: For any real hypersurface M of an almost hyperbolic Hermitian manifold the induced hyperdistribution D of M with almost hyperbolic Hermitian structure J is always torsionless[6-14].

Equations (3.26) imply

Proposition 3.2: For any real hypersurface M of a almost hyperbolic Hermitian manifold, the Levi-tensor G of an almost para contact structure (f, ξ, θ) induced in M has components of the form

(3.29) $G_{cb} = h_{cb} + f_c^e\ f_b^d\ h_{ed}$ (mod θ_c, θ_b)

in U, when an affine normal C to M is given in a coordinate neighborhood U of M. Proposition 3.2 implies that g(X, Y) = g(Y, X), g (JX, JY) + g(X, Y) = 0
for any vector fields X and Y belonging to the hyperdistribution D with hyperbolic RAC Structure J. Equations (3.25) imply

Proposition 3.3: Let (f, ξ, θ) be an almost para contact structure induced on a real hypersurface M of an almost hyperbolic Hermitian manifold by giving an affine normal C to M in a coordinate neighborhood U of M. Then (f, ξ, θ) is normal if and only if

(3.30) $-H_e^a f_b^e + f_e^a H_b^e - l_b \xi^a \equiv 0$, (mod θ_b).

We take another affine normal $\overline{C}$ to M in U and assume $\overline{C}$ is given by (2.6). Denote by $\overline{\nabla}, \overline{l}, \overline{h}$ and $\overline{H}$ respectively the induced affine connection, the third fundamental tensor, the covariant and the mixed second fundamental tensors of M in U, which are determined by (3.6), (3.14), (3.9) and (3.12) in term s of $\overline{C}$.

Then components $\overline{T}^{b}_{ca}$ *of* $\overline{\nabla}$, $\overline{h}_{cb}$ *of* $\overline{h}, \overline{H}^{a}_{b}$ *of* $\overline{H}$ *and* $\overline{l}_b$ *of* $\overline{l}$ are respectively given by

$$\overline{T}^{a}_{cb} = T^{a}_{cb} + h_{cb} A^{a}, \qquad \overline{h}_{cb} = \alpha\, h_{cb},$$

(3.31) $$\overline{H}^{a}_{b} = -\frac{1}{\alpha}[H^{a}_{b} + \nabla_b A^{a} - (l_b + h_{be} A^{e}) A^{a}],$$

$$\overline{l}_b = -(l_b + h_{be} A^{e}) + \nabla_b \log|\alpha| ,$$

where α is a non-vanishing function and $A = A^{a} B^{h}_{a}\, \partial / \partial x^{h}$ is a vector field belonging to D, both being defined in U. To obtain (3.31), we have used (2.6), $\overline{B}^{h}_{b} = B^{h}_{b}$ and

(3.32) $$\overline{B}^{a}_{i} = B^{a}_{i} + A^{a} C_i, \qquad \overline{c}_i = C_i,$$

where $\begin{pmatrix} \overline{B}^{a}_{i} \\ \overline{C}_i \end{pmatrix} = -(\overline{B}^{h}_{a}, \overline{C}^{h})^{-1}$.

Proposition 3.2 and $\overline{h}_{cb} = \alpha\, h_{cb}$ appearing in (3.21) imply the following well known Proposition [1, 2, 3, 5]:

Proposition 3.4. Let (f, ξ, θ) and $(\overline{f}, \overline{\xi}, \overline{\theta})$ be two almost para contact structures induced on a real hypersurface M and assume that they are related to each other by (2.7). Then $\overline{G}_{cb} = \alpha\, G_{cb}$ (mod θ_c, θ_b), α being a non-vanishing function, where $\overline{G}_{cb}$ *and* G_{cb} are respectively components of the Levi tensors of (f, ξ, θ) and $(\overline{f}, \overline{\xi}, \overline{\theta})$.

Proposition 3.4 shows that the restriction G_D of the Levi tensor G to D is determined up to a non-vanishing factor. Thus G_D is sometimes called the Levi tensor of the induced hyperdistribution D with almost hyperbolic Hermitian structure. When G is of the maximum rank 2n everywhere in M, the real hypersurface M is said to be non-degenerate. By P3.1, for any real hypersurface M of an almost hyperbolic Hermitian manifold the hyperdistribution D of M with almost hyperbolic Hermitian structure is torsionless. This fact means that any real hypersurface M admits a pseudo-conformal structure when M is non-degenerate[1,2,5].

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