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# Polynomials Related to Generalized Fibonacci Sequence

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**Abstract**

In this paper we extend the Fibonacci recurrence relation to define the sequence $\{C_n\}$ and derive some properties of this sequence. We also define the four comparison sequence $\{P_n\}$, $\{Q_n\}$, $\{R_n\}$, $\{S_n\}$. We also obtain some identities with the help of generating matrix.

*Key words:* Fibonacci Number, Fibonacci Sequence

**Mathematics Subject Classification:** 11B39

## 1 Introduction

Waddili, M.E.[10] has extended the Fibonacci recurrence relation to define the sequence $\{K_n\}$, where, $K_n = K_{n-1} + K_{n-2} + K_{n-3}, \quad n \geq 3$ (1.1)

and $K_0$, $K_1$, $K_2$ are given arbitrary algebraic integers.

Jaiswal, D.V.[8] has extended Fibonacci recurrence relation to define the sequence $\{Q_n\}$, where, $Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4}, \quad n \geq 4$ (1.2)

and, $Q_0$, $Q_1$, $Q_2$ are given arbitrary algebraic integers.

Harne, S.[7] has extended Fibonacci recurrence relation to define the sequence $\{D_n\}$, where,

$$D_n = D_{n-1} + D_{n-2} + D_{n-3} + D_{n-4} + D_{n-5}, \quad n \geq 5 \tag{1.3}$$

and, $D_0$, $D_1$, $D_2$ are given arbitrary algebraic integers.

In this paper we shall further extend the Fibonacci recurrence relation[3-6] to define the sequence $\{C_n\}$ and shall discuss some properties of this sequence. We shall also consider the four comparison sequence $\{P_n\}$, $\{Q_n\}$, $\{R_n\}$, $\{S_n\}$.

*2 The Generalized Sequence $\{C_n\}$ :*

We consider the sequence, $\{C_n\} = C_0, C_1, C_2, C_3, \ldots, C_n$

where, $C_0$, $C_1$, $C_2$, $C_3$, $C_4$, $C_5$ are arbitrary algebraic integers all of which are not zero and

$$C_n = C_{n-1} + C_{n-2} + C_{n-3} + C_{n-4} + C_{n-5} + C_{n-6}, \quad n \geq 6 \tag{2.1}$$

We also consider the sequence $\{P_n\} = P_0, P_1, P_2, \ldots, P_n$

where, $P_0 = C_3 - C_2 - C_1 - C_0$, $P_1 = C_4 - C_3 - C_2 - C_1$, $P_2 = C_5 - C_4 - C_3 - C_2$,

$$P_3 = C_6 - C_5 - C_4 - C_3, \; P_4 = C_7 - C_6 - C_5 - C_4 \tag{2.2}$$

with, $$P_n = C_{n-1} + C_{n-2} + C_{n-3} + C_{n-4} + C_{n-5}, \quad n \geq 5 \tag{2.3}$$

and $\{Q_n\} = Q_0, Q_1, Q_2, \ldots, Q_n,$ where, $Q_0 = C_4 - C_3 - C_2 - C_1 - C_0$,

$$Q_1 = C_5 - C_4 - C_3 - C_2 - C_1, \quad Q_2 = C_6 - C_5 - C_4 - C_3 - C_2, \tag{2.4}$$

with, $$Q_n = C_{n-1} + C_{n-2} + C_{n-3} + C_{n-4}, \tag{2.5}$$

and $\{R_n\} = R_0, R_1, R_2, \ldots, R_n$

where, $R_0 = C_5 - C_4 - C_3 - C_2 - C_1 - C_0$, $R_1 = C_6 - C_5 - C_4 - C_3 - C_2 - C_1$

$R_2 = C_7 - C_6 - C_5 - C_4 - C_3 - C_2$, $R_3 = C_8 - C_7 - C_6 - C_5 - C_4 - C_3$

$R_4 = C_9 - C_8 - C_7 - C_6 - C_5 - C_4$

with, $$R_n = C_{n-1} + C_{n-2} + C_{n-3}, \quad n \geq 3 \tag{2.6}$$

and $$\{S_n\} = S_0, S_1, S_2, \ldots, S_n \tag{2.7}$$

where, $S_0 = C_6 - C_5 - C_4 - C_3 - C_2 - C_1 - C_0$, $S_1 = C_7 - C_6 - C_5 - C_4 - C_3 - C_2 - C_1$

$S_2 = C_8 - C_7 - C_6 - C_5 - C_4 - C_3 - C_2$, $S_3 = C_9 - C_8 - C_7 - C_6 - C_5 - C_4 - C_3$

$$S_4 = C_{10} - C_9 - C_8 - C_7 - C_6 - C_5 - C_4 \tag{2.8}$$

with, $$S_n = C_{n-1} + C_{n-2}, \quad n \geq 2 \tag{2.9}$$

From (2.1) and (2.3) we have for $n \geq 11$

$$\begin{aligned} P_n = {} & C_{n-2} + C_{n-3} + C_{n-4} + C_{n-5} + C_{n-6} + C_{n-7} + C_{n-3} + C_{n-4} + C_{n-5} + C_{n-6} + C_{n-7} + C_{n-8} \\ & + C_{n-4} + C_{n-5} + C_{n-6} + C_{n-7} + C_{n-8} + C_{n-9} + C_{n-5} + C_{n-6} + C_{n-7} + C_{n-8} + C_{n-9} + C_{n-10} \\ & + C_{n-6} + C_{n-7} + C_{n-8} + C_{n-9} + C_{n-10} + C_{n-11} \end{aligned}$$

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5} + P_{n-6}$$

Now, from equation (2.2) and (2.3),

$P_{10} = (C_8 + C_7 + C_6 + C_5 + C_4) + (C_7 + C_6 + C_5 + C_4 + C_3)+ (C_6 + C_5 + C_4 + C_3 + C_2) + (C_5 + C_4 + C_3 + C_2 + C_1)+ (C_4 + C_3 + C_2 + C_1 + C_0) + (C_7 - C_6 - C_5 - C_4)$

$P_{10} = P_9 + P_8 + P_7 + P_6 + P_5 + P_4$

Similarly, $P_9 = P_8 + P_7 + P_6 + P_5 + P_4 + P_{3,}$ $P_8 = P_7 + P_6 + P_5 + P_4 + P_3 + P_2$

$P_7 = P_6 + P_5 + P_4 + P_3 + P_2 + P_1$

Hence, we have for $n \geq 6$ $P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5} + P_{n-6}$ (2.10)

Proceeding on similar lines, it can be shown that for $n \geq 6$.

$Q_n = C_{n-2} + C_{n-3} + C_{n-4} + C_{n-5} + C_{n-6} + C_{n-7}+ C_{n-3} + C_{n-4} + C_{n-5} + C_{n-6} + C_{n-7} + C_{n-8} + C_{n-4} + C_{n-5} + C_{n-6} + C_{n-7} + C_{n-8} + C_{n-9}+ C_{n-5} + C_{n-6} + C_{n-7} + C_{n-8} + C_{n-9} + C_{n-10}$

$Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4} + Q_{n-5} + Q_{n-6}$ for $n \geq 6$ (2.11)

Proceeding on similar lines it can be shown that for $n \geq 6$

$R_n = C_{n-2} + C_{n-3} + C_{n-4} + C_{n-5} + C_{n-6} + C_{n-7}+ C_{n-3} + C_{n-4} + C_{n-5} + C_{n-6} + C_{n-7} + C_{n-8} + C_{n-4} + C_{n-5} + C_{n-6} + C_{n-7} + C_{n-8} + C_{n-9}$

$R_n = R_{n-1} + R_{n-2} + R_{n-3} + R_{n-4} + R_{n-5} + R_{n-6}$ for $n \geq 6$ (2.12)

Proceeding on similar lines it can be shown that for $n \geq 6$

$S_n = C_{n-2} + C_{n-3} + C_{n-4} + C_{n-5} + C_{n-6} + C_{n-7}+ C_{n-3} + C_{n-4} + C_{n-5} + C_{n-6} + C_{n-7} + C_{n-8}$

$S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5} + S_{n-6}$ for $n \geq 6$ (2.13)

Thus, the four sequences $\{P_n\}$, $\{Q_n\}$, $\{R_n\}$ and $\{S_n\}$ are special cases of sequence $\{C_n\}$ and all obtained by taking different initial values [1,2,9].

On taking, $C_0 = C_1 = C_2 = 0, C_3 = C_4 = 1, C_5 = 2, C_0 = C_1 = 0$, C2 = 1, $C_3 = 0, C_4 = 1, C_5 = 2, C_0 = 0, C_1 = 1, C_2 = C_3 = 0, C_4 = 1, C_5 = 2, C_0 = 1, C_1 = C_2 = C_3 = 0, C_4 = 1, C_5 = 2, C_0 = C_2 = C_2 = C_3 = 0, C_4 =1, C_5 = 2$ (2.14)

$0,0,0,1,1,2,4,8,16,32,63,\ldots J_n,\ldots$

$0,0,1,0,1,2,4,8,16,31,62,\ldots K_n,\ldots$

$0,1,0,0,1,2,4,8,15,30,59,\ldots L_n,\ldots$

$1,0,0,0,1,2,4,7,14,28,56,\ldots M_n,\ldots$

$0,0,0,0,1,2,3,6,12,24,48,\ldots N_n,\ldots$

Here, we find that

$K_n = J_{n-1} + J_{n-2} + J_{n-3} + J_{n-4} + J_{n-5,}$ $L_n = J_{n-1} + J_{n-2} + J_{n-3} + J_{n-4}$

$M_n = J_{n-1} + J_{n-2} + J_{n-3,}$ $N_n = J_{n-1} + J_{n-2}$

Hence, we say that $\{J_n\}$ is $C_n$ type sequence, while $\{K_n\}$ is $P_n$ type sequence, and $\{L_n\}$ is $Q_n$

type sequence, while $\{M_n\}$ is $R_n$ type sequence, and $\{N_n\}$ is $S_n$ type sequence.

*3 Linear Sums And Some Properties* We have derived simple properties of the sequence $\{C_n\}$, $\{P_n\}$, $\{Q_n\}$, $\{R_n\}$, $\{S_n\}$ expressing each of the terms $C_6, C_7, C_8, \ldots, C_{n+5}$ as the sum of its six preceding terms as given in (2.1) adding both sides we obtained on simplification –

$$\sum_{i=0}^{n} C_i = 1/5\ \{C_{n+5} - C_{n+3} - 2C_{n+2} - 3C_{n+1} + C_n - (C_5 - C_3 - 2C_2 - 3C_1 - 4C_0)\ \} \tag{3.1}$$

On using (2.1), (2.2), (2.4), (2.6) and (2.8), we get

$$\sum_{i=0}^{n} C_{6i} = \sum_{i=0}^{6n-1} C_i + C_0, \sum_{i=0}^{n} C_{6i+2} = \sum_{i=0}^{6n+1} C_i + P_0, \sum_{i=0}^{n} C_{6i+3} = \sum_{i=0}^{6n+2} C_i + Q_0$$

$$\sum_{i=0}^{n} C_{6i+4} = \sum_{i=0}^{6n+3} C_i + R_0, \sum_{i=0}^{n} C_{6i+5} = \sum_{i=0}^{6n+4} C_i + S_{0,}\ \sum_{i=0}^{n} C_{6i+6} = \sum_{i=0}^{6n+5} C_i + (S_1 - C_0)$$

$$\sum_{i=0}^{n} C_{6i+5} = \sum_{i=0}^{6n+4} C_i + (R_1 - C_0), \sum_{i=0}^{n} C_{6i+4} = \sum_{i=0}^{6n+3} C_i + (Q_{-1} - C_0),\quad \sum_{i=0}^{n} C_{6i+3} = \sum_{i=0}^{6n+2} C_i + (P_1 - C_0)$$

*4 Property of Sequence $\{J_n\}$*

**Theorem**: For the sequence $\{J_n\}$ we have,

$$\begin{vmatrix} J_n & J_{n+1} & J_{n+2} & J_{n+3} & J_{n+4} & J_{n+5} \\ J_{n+1} & J_{n+2} & J_{n+3} & J_{n+4} & J_{n+5} & J_{n+6} \\ J_{n+2} & J_{n+3} & J_{n+4} & J_{n+5} & J_{n+6} & J_{n+7} \\ J_{n+3} & J_{n+4} & J_{n+5} & J_{n+6} & J_{n+7} & J_{n+8} \\ J_{n+4} & J_{n+5} & J_{n+6} & J_{n+7} & J_{n+8} & J_{n+9} \\ J_{n+5} & J_{n+6} & J_{n+7} & J_{n+8} & J_{n+9} & J_{n+10} \end{vmatrix} = (-1)^{n+1} \tag{4.1}$$

Proof : Consider the determinant –

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}, \quad \Delta^2 = \begin{vmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

Now, by mathematical induction,

$$\Delta^n = \begin{vmatrix} J_{n+1} & K_{n+1} & L_{n+1} & M_{n+1} & N_{n+1} & J_n \\ J_n & K_n & L_n & M_n & N_n & J_{n-1} \\ J_{n-1} & K_{n-1} & L_{n-1} & M_{n-1} & N_{n-1} & J_{n-2} \\ J_{n-2} & K_{n-2} & L_{n-2} & M_{n-2} & N_{n-2} & J_{n-3} \\ J_{n-3} & K_{n-3} & L_{n-3} & M_{n-3} & N_{n-3} & J_{n-4} \\ J_{n-4} & K_{n-4} & L_{n-4} & M_{n-4} & N_{n-4} & J_{n-5} \end{vmatrix}$$

Now, writing $N_{n+1} = J_n + J_{n-1}$ the R.H.S. can be written as the sum of two determinants, one of which is zero, Therefore,

$$\Delta^n = \begin{vmatrix} J_{n+1} & K_{n+1} & L_{n+1} & M_{n+1} & J_{n-1} & J_n \\ J_n & K_n & L_n & M_n & J_{n-2} & J_{n-1} \\ J_{n-1} & K_{n-1} & L_{n-1} & M_{n-1} & J_{n-3} & J_{n-2} \\ J_{n-2} & K_{n-2} & L_{n-2} & M_{n-2} & J_{n-4} & J_{n-3} \\ J_{n-3} & K_{n-3} & L_{n-3} & M_{n-3} & J_{n-5} & J_{n-4} \\ J_{n-4} & K_{n-4} & L_{n-4} & M_{n-4} & J_{n-6} & J_{n-5} \end{vmatrix}$$

Now, writing $M_{n+1} = J_n + J_{n-1} + J_{n-2}$, the R.H.S. can be written as the sum of three determinants, two of which are zero. Therefore,

$$\Delta^n = \begin{vmatrix} J_{n+1} & K_{n+1} & L_{n+1} & J_{n-2} & J_{n-1} & J_n \\ J_n & K_n & L_n & J_{n-3} & J_{n-2} & J_{n-1} \\ J_{n-1} & K_{n-1} & L_{n-1} & J_{n-4} & J_{n-3} & J_{n-2} \\ J_{n-2} & K_{n-2} & L_{n-2} & J_{n-5} & J_{n-4} & J_{n-3} \\ J_{n-3} & K_{n-3} & L_{n-3} & J_{n-6} & J_{n-5} & J_{n-4} \\ J_{n-4} & K_{n-4} & L_{n-4} & J_{n-7} & J_{n-6} & J_{n-5} \end{vmatrix}$$

Now, writing $L_{n+1} = J_n + J_{n-1} + J_{n-2} + J_{n-3}$, the R.H.S. can be written as the sum of four determinants, three of which are zero. Therefore,

$$\Delta^n = \begin{vmatrix} J_{n+1} & K_{n+1} & J_{n-3} & J_{n-2} & J_{n-1} & J_n \\ J_n & K_n & J_{n-4} & J_{n-3} & J_{n-2} & J_{n-1} \\ J_{n-1} & K_{n-1} & J_{n-5} & J_{n-4} & J_{n-3} & J_{n-2} \\ J_{n-2} & K_{n-2} & J_{n-6} & J_{n-5} & J_{n-4} & J_{n-3} \\ J_{n-3} & K_{n-3} & J_{n-7} & J_{n-6} & J_{n-5} & J_{n-4} \\ J_{n-4} & K_{n-4} & J_{n-8} & J_{n-7} & J_{n-6} & J_{n-5} \end{vmatrix}$$

Now, writing $K_{n+1} = J_n + J_{n-1} + J_{n-2} + J_{n-3} + J_{n-4}$ the R.H.S. can be written as the sum of five determinants, four of which are zero. Therefore,

$$\Delta^n = \begin{vmatrix} J_{n+1} & J_{n-4} & J_{n-3} & J_{n-2} & J_{n-1} & J_n \\ J_n & J_{n-5} & J_{n-4} & J_{n-3} & J_{n-2} & J_{n-1} \\ J_{n-1} & J_{n-6} & J_{n-5} & J_{n-4} & J_{n-3} & J_{n-2} \\ J_{n-2} & J_{n-7} & J_{n-6} & J_{n-5} & J_{n-4} & J_{n-3} \\ J_{n-3} & J_{n-8} & J_{n-7} & J_{n-6} & J_{n-5} & J_{n-4} \\ J_{n-4} & J_{n-9} & J_{n-8} & J_{n-7} & J_{n-6} & J_{n-5} \end{vmatrix}$$

On arranging, we get

$$\Delta^n = \begin{vmatrix} J_{n+1} & J_n & J_{n-1} & J_{n-2} & J_{n-3} & J_{n-4} \\ J_n & J_{n-1} & J_{n-2} & J_{n-3} & J_{n-4} & J_{n-5} \\ J_{n-1} & J_{n-2} & J_{n-3} & J_{n-4} & J_{n-5} & J_{n-6} \\ J_{n-2} & J_{n-3} & J_{n-4} & J_{n-5} & J_{n-6} & J_{n-7} \\ J_{n-3} & J_{n-4} & J_{n-5} & J_{n-6} & J_{n-7} & J_{n-8} \\ J_{n-4} & J_{n-5} & J_{n-6} & J_{n-7} & J_{n-8} & J_{n-9} \end{vmatrix}$$

Putting, n-9 = m or n = m+9 and substituting all the Δ's, we obtain,

$$(-1)^{m+9} = \begin{vmatrix} J_{m+10} & J_{m+9} & J_{m+8} & J_{m+7} & J_{m+6} & J_{m+5} \\ J_{m+9} & J_{m+8} & J_{m+7} & J_{m+6} & J_{m+5} & J_{m+4} \\ J_{m+8} & J_{m+7} & J_{m+6} & J_{m+5} & J_{m+4} & J_{m+3} \\ J_{m+7} & J_{m+6} & J_{m+5} & J_{m+4} & J_{m+3} & J_{m+2} \\ J_{m+6} & J_{m+5} & J_{m+4} & J_{m+3} & J_{m+2} & J_{m+1} \\ J_{m+5} & J_{m+4} & J_{m+3} & J_{m+2} & J_{m+1} & J_m \end{vmatrix}$$

Rearranging the determinant and replace m by n we get the required result (4.1).

*5 Generating Matrix $\{C_n\}$ :*

Now, we obtain some identities with the help of generating matrix, we consider the matrix,

$$[T] = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix} \tag{5.1}$$

By mathematical induction we can show that –

$$[T]^n = \begin{vmatrix} J_{n+1} & K_{n+1} & L_{n+1} & M_{n+1} & N_{n+1} & J_n \\ J_n & K_n & L_n & M_n & N_n & J_{n-1} \\ J_{n-1} & K_{n-1} & L_{n-1} & M_{n-1} & N_{n-1} & J_{n-2} \\ J_{n-2} & K_{n-2} & L_{n-2} & M_{n-2} & N_{n-2} & J_{n-3} \\ J_{n-3} & K_{n-3} & L_{n-3} & M_{n-3} & N_{n-3} & J_{n-4} \\ J_{n-4} & K_{n-4} & L_{n-4} & M_{n-4} & N_{n-4} & J_{n-5} \end{vmatrix} \quad \text{where, } n \geq 5 \tag{5.2}$$

$$[C_n, C_{n-1}, C_{n-2}, C_{n-3}, C_{n-4}, C_{n-5}] = [T]^{n-5}\,[C_5, C_4, C_3, C_2, C_1, C_0\,] \tag{5.3}$$

On using (5.2) and (5.3), we get –

$$\begin{bmatrix} C_{n+P} \\ C_{n+P-1} \\ C_{n+P-2} \\ C_{n+P-3} \\ C_{n+P-4} \\ C_{n+P-5} \end{bmatrix} = \begin{bmatrix} J_{n+1} & K_{n+1} & L_{n+1} & M_{n+1} & N_{n+1} & J_n \\ J_n & K_n & L_n & M_n & N_n & J_{n-1} \\ J_{n-1} & K_{n-1} & L_{n-1} & M_{n-1} & N_{n-1} & J_{n-2} \\ J_{n-2} & K_{n-2} & L_{n-2} & M_{n-2} & N_{n-2} & J_{n-3} \\ J_{n-3} & K_{n-3} & L_{n-3} & M_{n-3} & N_{n-3} & J_{n-4} \\ J_{n-4} & K_{n-4} & L_{n-4} & M_{n-4} & N_{n-4} & J_{n-5} \end{bmatrix} \begin{bmatrix} C_n \\ C_{n-1} \\ C_{n-2} \\ C_{n-3} \\ C_{n-4} \\ C_{n-5} \end{bmatrix}$$

From this we obtain –

$$C_{n+P} = J_{P+1}D_n + K_{P+1}D_{n-1} + L_{P+1}D_{n-2} + M_{P+1}D_{n-3} + N_{P+1}D_{n-4} + J_nD_{n-5} \tag{5.4}$$

Let us now consider the matrix [W] which is transpose of the matrix [T] in,

$$[W] = [T]' = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$ It can be shown that the sequence,

$$C_4, P_5, Q_5, R_5, S_5, C_5, \ldots, C_{n-1}, P_n, Q_n R_n, S_n, C_n \tag{5.5}$$

It is generated by matrix [W]

$$[C_n, P_n, Q_n, R_n, S_n, C_{n-1}] = [W]^{n-5}\,[C_5, P_5, Q_5, R_5, S_5, C_4],\ \ n \geq 5 \tag{5.6}$$

On using (5.5) and (5.6), we get $[C_{n+P}, P_{n+P}, Q_{n+P}, R_{n+P}, S_{n+P}, C_{n+P-1}]$

$$= [W]^{n+P-5}\,[C_5, P_5, Q_5, R_5, S_5, C_4], \qquad n \geq 5$$

$$= [W]^{P}\quad [C_n, P_n, Q_n, R_n, S_n, C_{n-1}]$$

$$= \begin{bmatrix} J_{P+1} & J_P & J_{P-1} & J_{P-2} & J_{P-3} & J_{P-4} \\ K_{P+1} & K_P & K_{P-1} & K_{P-2} & K_{P-3} & K_{P-4} \\ L_{P+1} & L_P & L_{P-1} & L_{P-2} & L_{P-3} & L_{P-4} \\ M_{P+1} & M_P & M_{P-1} & M_{P-2} & M_{P-3} & M_{P-4} \\ N_{P+1} & N_P & N_{P-1} & N_{P-2} & N_{P-3} & N_{P-4} \\ J_P & J_{P-1} & J_{P-2} & J_{P-3} & J_{P-4} & J_{P-5} \end{bmatrix} \begin{bmatrix} C_n \\ P_n \\ Q_n \\ R_n \\ S_n \\ C_{n-1} \end{bmatrix}$$

$$C_{n+P} = J_{P+1}C_n + J_P P_n + J_{P-1}Q_n + J_{P-2}R_n + J_{P-3}S_n + J_{P-4}C_{n-1}$$

$$P_{n+P} = K_{P+1}C_n + K_P P_n + K_{P-1}Q_n + K_{P-2}R_n + K_{P-3}S_n + K_{P-4}C_{n-1}$$

$$Q_{n+P} = L_{P+1}C_n + L_P P_n + L_{P-1}Q_n + L_{P-2}R_n + L_{P-3}S_n + L_{P-4}C_{n-1}$$

$$R_{n+P} = M_{P+1}C_n + M_P P_n + M_{P-1}Q_n + M_{P-2}R_n + M_{P-3}S_n + M_{P-4}C_{n-1}$$

$$S_{n+P} = N_{P+1}C_n + N_P P_n + N_{P-1}Q_n + N_{P-2}R_n + N_{P-3}S_n + N_{P-4}C_{n-1}$$

## Conclusion and its application

There are many known identities for Fibonacci recursion relation. We define the sequence $\{C_n\}$ and its four comparison sequence $\{P_n\}$, $\{Q_n\}$, $\{R_n\}$, $\{S_n\}$.We derive linear sum and properties of comparison sequence. We also derive generating matrix for $\{C_n\}$.

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**Scope of the future work**

More than four comparison sequence can be defined and new relationship have been derived.

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